Quantum mechanics for the Math-minded

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Sundays 8pm
Tuesdays and Thursdays 8:30pm

Psets - erratic, leftovers from class
+ warm-up psets

Final paper due a week after class ends
~5 pages

Final presentation - during the last week
~25 minutes (or ~50 minutes)

Here's a sentence we want to understand:

A quantum system is a finite-dimension complex inner-product space.

A quantum computer is a quantum system with underlying space $\mathbb{C}^{2^n}$

We say the qc. has n-qubits if the dimension of its system is 2^n.

Humans can interact with quantum computers/systems in 2 ways:

- We can give the computer instructions in the form of gates, or equiv. with unitary maps

- We can ask the computer a question about its state.

this is called a measurement

Def: Given a complex vector space V, an inner product on V is a function $V \times V \to \mathbb{C}$

$$(v, w) \mapsto \langle v, w \rangle$$

satisfying:

1) $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$

$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$

2) If $\lambda \in \mathbb{C}$,

$$\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$$

$$\langle \lambda v, w\rangle = \lambda^* \langle v, w\rangle$$

3) $\langle v, w\rangle = (\langle w, v\rangle)^*$

4) If $v \neq 0$, then $\langle v, v\rangle > 0$.

Example: If $V = \mathbb{C}^n$, the standard inner product on V is given by

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Exercise:

What is $\langle ae_1 + be_2, ce_1 + de_2\rangle$?

$\langle ae_1, ce_1\rangle + \sim + \langle be_2, de_2\rangle$

$a^* c + b^* d$

$\langle v, w \rangle = v_1^* w_1 + \cdots + v_n^* w_n$

Def: An orthonormal basis of an inner product space V is a basis $e_1, \ldots, e_n$ such that $\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Problem: Show that there is an orthonormal basis for any inner product on $\mathbb{C}^2$.

This can be done for $\mathbb{C}^n$

Gram-Schmidt orthonormalization

Bra-ket notation

A ket is something like $|v\rangle$

$$\underbrace{|}_{\text{notation}}\ \underbrace{v}_{\text{vector}}\ \underbrace{\rangle}_{\text{notation}}$$

$|v\rangle$ means the same thing as v

(when v is a vector)

Given a linear functional $f: V \to \mathbb{C}$

then we write a bra $\langle f|$

to denote that functional.

When we combine a bra and a ket

we get a complex number

e.g. $\langle f | v \rangle$ denotes the

number $f(v) \in \mathbb{C}$

We can also put vectors in bras.

Then $\langle v | w \rangle$ denotes $\langle v, w \rangle \in \mathbb{C}$

This is consistent with the functional

notation, since if we fix

$v \in V$, the map $w \mapsto \langle v, w \rangle$

is a linear functional $V \to \mathbb{C}$

If $L : V \to V$ is a linear map,

then we can write $L | v \rangle$

to denote $L(v)$

$\rightsquigarrow \quad L|v\rangle = |Lv\rangle$

We can also put bras on these expressions

e.g $\langle v | L | w\rangle = \langle v, L(w)\rangle$

we can always write $\langle Lv | w\rangle$

for $\langle L(v), w\rangle$.

Def: Given inner product spaces V, W

a unitary map $\mathcal{U}: V \to W$

is one that satisfies

$$\langle \mathcal{U}v, \mathcal{U}w\rangle_W = \langle v, w\rangle_V$$

i.e. a unitary map preserves inner products

Exercise: Check that U is unitary iff it takes an orthonormal basis for V to an orthonormal set of vectors in W.

In particular, if we use the standard inner product on $\mathbb{C}^n$, then the matrix of $U: \mathbb{C}^n \to \mathbb{C}^n$ is unitary if the columns are an orthonormal basis on $\mathbb{C}^n$.

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Note: An orthonormal set of vectors is always linearly independent. Why?

The group of unitary maps $\mathbb{C}^n \to \mathbb{C}^n$ is called the unitary group $U(n)$.
This is important for representation theory.

Soln to Problem 3:

Unitary maps $\mathbb{C}^n \to \mathbb{C}^n$ are closed under composition

$$\langle U_1 U_2 v, U_1 U_2 w \rangle$$
$$= \langle U_2 v, U_2 w \rangle$$
$$= \langle v, w \rangle.$$

identity is unitary ✓

inverse: the $\ker U = 0$
because $\langle Uv, Uv \rangle = \langle v, v \rangle > 0$

whenever $v \neq 0$, so $Uv \neq 0$.

$\Rightarrow$ Since $\mathbb{C}^n$ and $\mathbb{C}^n$ have the same dimension, a linear map with kernel 0 is invertible w/ inverse U^{-1}

$$\langle U^{-1}Uv, U^{-1}Uw \rangle = \langle v, w \rangle$$
$$= \langle Uv, Uw \rangle$$

$\Rightarrow$ U^{-1} is unitary